INTERSECTION THEORY FOR ERGODIC SOLENOIDS

VICENTE MUÑOZ AND RICARDO PÉREZ MARCO

Abstract. We develop the intersection theory associated to immersed, oriented and measured solenoids, which were introduced in [1].

1. Introduction

In [1], the authors define the concept of k-solenoid as an abstract laminated space, and prove that an solenoid with a transversal measure immersed in a smooth manifold defines a generalized Ruelle-Sullivan current. The purpose of the current paper is to study the intersection theory of such objects.

If M is a smooth manifold, any closed oriented submanifold $N \subset M$ of dimension k determines a homology class in $H_k(M, \mathbb{Z})$. This homology class in $H_k(M, \mathbb{R})$, as dual of De Rham cohomology, is explicitly given by integration of the restriction to N of differential k-forms on M. For representing real homology classes, we need to consider more general objects. In [1], we define a k-solenoid to be a Hausdorff compact space foliated by k-dimensional leaves with finite dimensional transversal structure (see the precise definition in section 2). For these oriented solenoids we can consider k-forms that we can integrate provided that we are given a transversal measure invariant by the holonomy group. We define an immersion of a solenoid S into M to be a regular map $f : S \to M$ that is an immersion on each leaf. If the solenoid S is endowed with a transversal measure μ, then any smooth k-form in M can be pulled back to S by f and integrated. This defines a closed current that we denote by (f, S_μ) and call a generalized current. This defines a homology class $[f, S_\mu] \in H_k(M, \mathbb{R})$. This construction generalizes the currents introduced by Ruelle and Sullivan in [6].

In [3], we prove that every real homology class in $H_k(M, \mathbb{R})$ can be realized by a generalized current (f, S_μ) where S_μ is an oriented, minimal, uniquely ergodic solenoid. Uniquely ergodic solenoids are defined in [1]. These are minimal solenoids which possess a unique transversal measure. The importance of such solenoids stems from the fact that their topology, more precisely the recurrence of any leaf, determines its solenoidal structure. This is formulated in a precise way through the Schwartzman measures in [2]. Finally, in [4] we prove that the generalized currents associated to oriented, minimal, uniquely ergodic, immersed solenoids are dense in the space of closed currents.

Let us review the contents of the paper. In section 2 we recall the definition of solenoids and of the generalized current associated to an immersed, oriented and measured solenoid. In section 3 we prove that the generalized current is invariant by perturbation (homotopy) of a solenoid. Section 4 is devoted to defining the intersection of two solenoids S_{1,μ_1}, S_{2,μ_2} which intersect transversally (i.e. when the leaves intersect transversally). In this case, the intersection is a solenoid and it has a natural transversal measure associated to the measures

2000 *Mathematics Subject Classification.* Primary: 37A99. Secondary: 58A25, 57R95, 55N45.
Key words and phrases. Real homology, Ruelle-Sullivan current, solenoid, ergodic, transversal intersection.
Partially supported through Spanish MEC grant MTM2007-63582.

of the two given solenoids. The generalized current of the intersection solenoid is the product of the generalized currents of the given solenoids. We also prove that we may homotop a solenoid to make it intersect transversally a submanifold whenever the transversal structure is Cantor.

In general, it is not possible to perturb solenoids to make them intersect transversally. This clearly holds in the case of solenoids. But it is also true when we try to perturb smoothly solenoids with transversal Cantor structure, as the persistence of homoclinic tangencies for stable and unstable foliations shows (see [5]). So the concept of almost everywhere transversality introduced in section 5 is very useful. In the case of complementary dimensions, two solenoids are said to intersect almost everywhere transversally if $\mu_1 \times \mu_2$-almost all leaves intersect transversally and the other leaves intersect in isolated points. We define a measure for the intersection and the integral of the measure (taking also into account the intersection index) equals the product of the generalized currents. In the case of non-complementary dimensions, we shall require that the models are conjugated to analytic solenoids. This allows to define an intersection current supported on the intersection of the solenoids. That is worked out in section 6.

Acknowledgements. The authors are grateful to Alberto Candel, Etienne Ghys, Nessim Sibony, Dennis Sullivan and Jaume Amorós for their comments and interest on this work. The first author wishes to acknowledge Universidad Complutense de Madrid and Institute for Advanced Study at Princeton for their hospitality and for providing excellent working conditions. The second author thanks Jean Bourgain and the IAS at Princeton for their hospitality and facilitating the collaboration of both authors.

2. Measured solenoids and generalized currents

Let us recall the definition of a k-solenoid from [1].

Definition 2.1. *Let $0 \leq s, r \leq \omega$, $r \geq s$, and let $k, l \geq 0$ be two integers. A foliated manifold (of dimension $k+l$, with k-dimensional leaves, of regularity $C^{r,s}$) is a smooth manifold W of dimension $k+l$ endowed with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$, $\varphi_i : U_i \to \mathbb{R}^k \times \mathbb{R}^l$, whose changes of charts*

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j) \ ,$$

are of the form $\varphi_{ij}(x, y) = (X_{ij}(x, y), Y_{ij}(y))$, where $Y_{ij}(y)$ is of class C^s and $X_{ij}(x, y)$ is of class $C^{r,s}$.

A flow-box for W is a pair (U, φ) consisting of an open subset $U \subset W$ and a map $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^l$ such that $\mathcal{A} \cup \{(U, \varphi)\}$ is still an atlas for W.

(Here C^ω are the analytic functions.)

Given two foliated manifolds W_1, W_2 of dimension $k+l$, with k-dimensional leaves, and of regularity $C^{r,s}$, a regular map $f : W_1 \to W_2$ is a continuous map which is locally, in flow-boxes, of the form $f(x, y) = (X(x, y), Y(y))$, where Y is of class C^s and X is of class $C^{r,s}$. A diffeomorphism $\phi : W_1 \to W_2$ is a homeomorphism such that ϕ and ϕ^{-1} are both regular maps.

Definition 2.2. **(k-solenoid)** *Let $0 \leq r \leq s \leq \omega$, and let $k, l \geq 0$ be two integers. A pre-solenoid of dimension k, of class $C^{r,s}$ and transversal dimension l is a pair (S, W) where W is a foliated manifold and $S \subset W$ is a compact subspace which is a collection of leaves.*

Two pre-solenoids (S, W_1) and (S, W_2) are equivalent if there are open subsets $U_1 \subset W_1$, $U_2 \subset W_2$ with $S \subset U_1$ and $S \subset U_2$, and a diffeomorphism $f : U_1 \to U_2$ such that f is the identity on S.

A k-solenoid of class $C^{r,s}$ and transversal dimension l (or just a k-solenoid, or a solenoid) is an equivalence class of pre-solenoids.

We usually denote a solenoid by S, without making explicit mention of W. We shall say that W defines the solenoid structure of S.

Definition 2.3. (Flow-box) *Let S be a solenoid. A flow-box for S is a pair (U, φ) formed by an open subset $U \subset S$ and a homeomorphism*

$$\varphi : U \to D^k \times K(U)\,,$$

where D^k is the k-dimensional open ball and $K(U) \subset \mathbb{R}^l$, such that there exists a foliated manifold W defining the solenoid structure of S, $S \subset W$, and a flow-box $\hat{\varphi} : \hat{U} \to \mathbb{R}^k \times \mathbb{R}^l$ for W, with $U = \hat{U} \cap S$, $\hat{\varphi}(U) = D^k \times K(U) \subset \mathbb{R}^k \times \mathbb{R}^l$ and $\varphi = \hat{\varphi}_{|U}$.

The set $K(U)$ is the transversal space of the flow-box. The dimension l is the transversal dimension.

As S is locally compact, any point of S is contained in a flow-box U whose closure $\overline{U}$ is contained in a bigger flow-box. For such flow-box, $\overline{U} \cong \overline{D}^k \times \overline{K}(U)$, where $\overline{D}^k$ is the closed unit ball, $\overline{K}(U)$ is some compact subspace of $\mathbb{R}^l$, and $U = D^k \times K(U) \subset \overline{D}^k \times \overline{K}(U)$. All flow-boxes that we shall use are of this type without making further explicit mention.

Definition 2.4. (Leaf) *A leaf of a k-solenoid S is a leaf l of any foliated manifold W inducing the solenoid structure of S, such that $l \subset S$. Note that this notion is independent of W.*

Definition 2.5. (Oriented solenoid) *An oriented solenoid is a solenoid S such that there is a foliated manifold $W \supset S$ inducing the solenoid structure of S, where W has oriented leaves (in a transversally continuous way).*

A solenoid is minimal if it does not contain a proper sub-solenoid. In [1] it is proven that minimal solenoids always exist.

Definition 2.6. (Transversal) *Let S be a k-solenoid. A local transversal at a point $p \in S$ is a subset T of S with $p \in T$, such that there is a flow-box (U, φ) of S with U a neighborhood of p containing T and such that*

$$\varphi(T) = \{0\} \times K(U)\,.$$

A transversal T of S is a compact subset of S such that for each $p \in T$ there is an open neighborhood V of p such that $V \cap T$ is a local transversal at p.

If S is a k-solenoid of class $C^{r,s}$, then any transversal T inherits an l-dimensional C^s-Whitney structure.

Definition 2.7. *A transversal T of S is a global transversal if all leaves intersect T.*

Definition 2.8. (Holonomy) *Given two points p_1 and p_2 in the same leaf, two local transversals T_1 and T_2, at p_1 and p_2 respectively, and a path $\gamma : [0,1] \to S$, contained in the leaf with endpoints $\gamma(0) = p_1$ and $\gamma(1) = p_2$, we define a germ of a map (the holonomy map)*

$$h_\gamma : (T_1, p_1) \to (T_2, p_2)\,,$$

by lifting γ to nearby leaves.

We denote by $\mathrm{Hol}_S(T_1, T_2)$ *the set of germs of holonomy maps from* T_1 *to* T_2. *These form the holonomy pseudo-group.*

Definition 2.9. (Transversal measure) *Let* S *be a* k*-solenoid. A transversal measure* $\mu = (\mu_T)$ *for* S *associates to any local transversal* T *a locally finite measure* μ_T *supported on* T*, which are invariant by the holonomy pseudogroup. More precisely, if* T_1 *and* T_2 *are two transversals and* $h : V \subset T_1 \to T_2$ *is a holonomy map, then*

$$h_*(\mu_{T_1}|_V) = \mu_{T_2}|_{h(V)} .$$

We assume that a transversal measure μ *is non-trivial, i.e. for some* T, μ_T *is non-zero.*

We denote by S_μ *a* k*-solenoid* S *endowed with a transversal measure* $\mu = (\mu_T)$. *We refer to* S_μ *as a measured solenoid.*

We fix now a C^∞ manifold M of dimension n.

Definition 2.10. (Immersion and embedding of solenoids) *Let* S *be a* k*-solenoid of class* $C^{r,s}$ *with* $r \geq 1$. *An immersion*

$$f : S \to M$$

is a regular map (that is, it has an extension $\hat{f} : W \to M$ *of class* $C^{r,s}$*, where* W *is a foliated manifold which defines the solenoid structure of* S*), such that the differential restricted to the tangent spaces of leaves has rank* k *at every point of* S. *We say that* $f : S \to M$ *is an immersed solenoid.*

Let $r, s \geq 1$. *A transversally immersed solenoid* $f : S \to M$ *is a regular map* $f : S \to M$ *such that it admits an extension* $\hat{f} : W \to M$ *which is an immersion (of a* $(k+l)$*-dimensional manifold into an* n*-dimensional one) of class* $C^{r,s}$*, such that the images of the leaves intersect transversally in* M.

An embedded solenoid $f : S \to M$ *is a transversally immersed solenoid of class* $C^{r,s}$*, with* $r, s \geq 1$*, with injective* f*, that is, the leaves do not intersect or self-intersect.*

Note that under a transversal immersion, resp. an embedding, $f : S \to M$, the images of the leaves are immersed, resp. injectively immersed, submanifolds.

Let M be a smooth manifold. We shall denote the space of compactly supported currents of dimension k by

$$\mathcal{C}_k(M) .$$

These currents are functionals $T : \Omega^k(M) \to \mathbb{R}$. A current $T \in \mathcal{C}_k(M)$ is closed if $T(d\alpha) = 0$ for any $\alpha \in \Omega^{k-1}(M)$. Therefore, by restricting to the closed forms, a closed current T defines a linear map

$$[T] : H^k(M, \mathbb{R}) \longrightarrow \mathbb{R} .$$

By duality, T defines a real homology class $[T] \in H_k(M, \mathbb{R})$.

Definition 2.11. (Generalized currents) *Let* S *be an oriented* k*-solenoid of class* $C^{r,s}$, $r \geq 1$*, endowed with a transversal measure* $\mu = (\mu_T)$. *An immersion*

$$f : S \to M$$

defines a current $(f, S_\mu) \in \mathcal{C}_k(M)$*, called generalized Ruelle-Sullivan current (or just generalized current), as follows. Let* ω *be an* k*-differential form in* M. *The pull-back* $f^*\omega$ *defines a* k*-differential form on the leaves of* S.

Let $S = \bigcup_i U_i$ be an open cover of the solenoid. Take a partition of unity $\{\rho_i\}$ subordinated to the covering $\{U_i\}$. We define

$$\langle (f, S_\mu), \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{L_y} \rho_i f^* \omega \right) d\mu_{K(U_i)}(y) \,,$$

where L_y denotes the horizontal disk of the flow-box.

The current (f, S_μ) is closed, hence it defines a real homology class

$$[f, S_\mu] \in H_k(M, \mathbb{R}) \,,$$

called Ruelle-Sullivan homology class.

From now on, we shall consider a C^∞ compact and oriented manifold M of dimension n. Let $f : S_\mu \to M$ be an oriented measured k-solenoid immersed in M. We shall denote

$$[f, S_\mu]^* \in H^{n-k}(M, \mathbb{R}) \,,$$

the dual of $[f, S_\mu]$ under the Poincaré duality isomorphism $H_k(M, \mathbb{R}) \cong H^{n-k}(M, \mathbb{R})$.

3. Homotopy of solenoids

Let us see that the Ruelle-Sullivan homology class defined by an immersed oriented measured k-solenoid does not change by perturbations.

Definition 3.1. (Solenoid with boundary) *Let $0 \leq s \leq r \leq \omega$, and let $k, l \geq 0$ be two integers. A foliated manifold with boundary (of dimension $k + l$, with k-dimensional leaves, of class $C^{r,s}$) is a smooth manifold W with boundary, of dimension $k + l$, endowed with an atlas $\{(U_i, \varphi_i)\}$ of charts*

$$\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^{k+l}_+ = \{(x_1, \ldots, x_k, y_1, \ldots, y_l \ ; \ x_1 \geq 0\} \,,$$

whose changes of charts are of the form $\varphi_i \circ \varphi_j^{-1}(x, y) = (X_{ij}(x, y), Y_{ij}(y))$, where $Y_{ij}(y)$ is of class C^s and $X_{ij}(x, y)$ is of class $C^{r,s}$.

A pre-solenoid with boundary is a pair (S, W) where W is a foliated manifold with boundary and $S \subset W$ is a compact subspace which is a collection of leaves.

Two pre-solenoids with boundary (S, W_1) and (S, W_2) are equivalent if there are open subsets $U_1 \subset W_1$, $U_2 \subset W_2$ with $S \subset U_1$ and $S \subset U_2$, and a diffeomorphism $f : U_1 \to U_2$ (preserving leaves, of class $C^{r,s}$) which is the identity on S.

A k-solenoid with boundary is an equivalence class of pre-solenoids with boundary.

Note that any manifold with boundary is a solenoid with boundary.

The boundary of a k-solenoid with boundary S is the $(k-1)$-solenoid (without boundary) ∂S defined by the foliated manifold ∂W, where W is a foliated manifold with boundary defining the solenoid structure of S.

A k-solenoid with boundary S has two types of flow-boxes. If $p \in S - \partial S$ is an interior point, then there is a flow-box (U, φ) with $p \in U$, of the form $\varphi : U \to D^k \times K(U)$. If $p \in \partial S$ is a boundary point, then there is a flow-box (U, φ) with $p \in U$ such that φ is a homeomorphism

$$\varphi : U \to D^k_+ \times K(U) \,,$$

where $D^k_+ = \{(x_1, \dots, x_k) \in D_k\,;\, x_1 \geq 0\}$, and $K(U) \subset \mathbb{R}^l$, $\varphi(p) = (0, \dots, 0, y_0)$, for some $y_0 \in K$. Note that writing

$$U' = \partial S \cap U = \varphi^{-1}(D^{k-1} \times K(U))\,,$$

where $D^{k-1} = \{(0, x_2, \dots, x_k) \in D_k\} \subset D^+_k$, we have that $(U', \varphi_{|U'})$ is a flow-box for ∂S. Therefore, if T is a transversal for ∂S, then it is also transversal for S.

For a solenoid with boundary S there is also a well-defined notion of holonomy pseudo-group. If T is a local transversal for ∂S, and $h : T \to T$ is a holonomy map for ∂S defined by a path in ∂S, then h lies in the holonomy pseudo-group of S. So

$$\mathrm{Hol}_{\partial S}(T) \subset \mathrm{Hol}_S(T)\,,$$

but they are in general not equal. In particular, if S is connected and minimal with non-empty boundary then

$$\mathcal{M}_{\mathcal{T}}(S) \subset \mathcal{M}_{\mathcal{T}}(\partial S)\,.$$

That is, if $\mu = (\mu_T)$ is a transversal measure for S, then it yields a transversal measure for ∂S, by considering only those transversals T which are transversals for ∂S. We denote this transversal measure by μ again.

If S comes equipped with an orientation, then ∂S has a natural induced orientation. Note that any leaf $l \subset S$ is a manifold with boundary and each connected component of ∂l is a leaf of ∂S.

Theorem 3.2. (Stokes theorem) *Let $f : S_\mu \to M$ be an oriented $(k+1)$-solenoid with boundary, endowed with a transversal measure, and immersed into a smooth manifold M. Let ω be a k-form on M. Then*

$$\langle [f, S_\mu], d\omega \rangle = \langle [f_{|\partial S}, \partial S_\mu], \omega \rangle\,.$$

Proof. Let $\{U_i\}$ be a covering of S by flow-boxes, and let $\{\rho_i\}$ be a partition of unity subordinated to it. Adding up the equalities

$$\begin{aligned} &\int_{K(U_i)} \left(\int_{L_y} d\rho_i \wedge f^*\omega \right) d\mu_{K(U_i)}(y) + \int_{K(U_i)} \left(\int_{L_y} \rho_i f^* d\omega \right) d\mu_{K(U_i)}(y) \\ &\qquad = \int_{K(U_i)} \left(\int_{L_y} d(\rho_i f^*\omega) \right) d\mu_{K(U_i)}(y) = \int_{K(U_i)} \left(\int_{\partial L_y} \rho_i f^*\omega \right) d\mu_{K(U_i)}(y)\,, \end{aligned}$$

for all i, and using that $\sum d\rho_i \equiv 0$, we get

$$\begin{aligned} \langle [f, S_\mu], d\omega \rangle &= \sum_i \int_{K(U_i)} \left(\int_{L_y} \rho_i f^* d\omega \right) d\mu_{K(U_i)}(y) \\ &= \sum_i \int_{K(U_i)} \left(\int_{\partial L_y} \rho_i f^*\omega \right) d\mu_{K(U_i)}(y) = \langle [f_{|\partial S}, \partial S_\mu], \omega \rangle\,. \end{aligned}$$

□

Let S be a k-solenoid of class $C^{r,s}$. We give $S \times I = S \times [0,1]$ a natural $(k+1)$-solenoid with boundary structure of the same class, by taking a foliated manifold $W \supset S$ defining the solenoid structure of S, and foliating $W \times I$ with the leaves $l \times I$, $l \subset W$ being a leaf of W. Then $S \times I \subset W \times I$ is a $(k+1)$-solenoid with boundary. The boundary of $S \times I$ is

$$(S \times \{0\}) \sqcup (S \times \{1\})\,.$$

If S is oriented then $S \times I$ is naturally oriented and its boundary consists of $S \times \{0\} \cong S$ with orientation reversed, and $S \times \{1\} \cong S$ with orientation preserved.

Moreover if T is a transversal for S, then $T' = T \times \{0\}$ is a transversal for $S' = S \times I$. The following is immediate.

Lemma 3.3. *There is an identification of the holonomies of S and $S \times I$. More precisely, under the identification $T \cong T' = T \times \{0\}$,*

$$\mathrm{Hol}_S(T) = \mathrm{Hol}_{S\times I}(T')\,.$$

In particular,

$$\mathcal{M}_{\mathcal{T}}(S) = \mathcal{M}_{\mathcal{T}}(S \times I)\,.$$

Definition 3.4. (Equivalence of immersions) *Two solenoid immersions $f_0 : S_0 \to M$ and $f_1 : S_1 \to M$ of class $C^{r,s}$ in M are immersed equivalent if there is a $C^{r,s}$-diffeomorphism $h : S_0 \to S_1$ such that*

$$f_0 = f_1 \circ h\,.$$

Two measured solenoid immersions are immersed equivalent if h can be chosen to preserve the transversal measures.

Definition 3.5. (Homotopy of immersions) *Let S be a k-solenoid of class $C^{r,s}$ with $r \geq 1$. A homotopy between immersions $f_0 : S \to M$ and $f_1 : S \to M$ is an immersion of solenoids $f : S \times I \to M$ such that $f_0(x) = f(x, 0)$ and $f_1(x) = f(x, 1)$.*

Definition 3.6. (Cobordism of solenoids) *Let S_0 and S_1 be two $C^{r,s}$-solenoids. A cobordism of solenoids is a $(k+1)$-solenoid S with boundary $\partial S = S_0 \sqcup S_1$.*

If S_0 and S_1 are oriented, then an oriented cobordism is a cobordism S which is an oriented solenoid such that it induces the given orientation on S_1 and the reversed orientation on S_0.

If S_0 and S_1 have transversal measures μ_0 and μ_1, respectively, then a measured cobordism is a cobordism S endowed with a transversal measure μ inducing the measures μ_0 and μ_1 on S_0 and S_1, respectively.

Definition 3.7. (Homology equivalence) *Let $f_0 : S \to M$ and $f_1 : S \to M$ be two immersed solenoids in M. We say that they are homology equivalent if there exists a cobordism of solenoids S between S_0 and S_1 and a solenoid immersion $f : S \to M$ with $f_{|S_0} = f_0$, $f_{|S_1} = f_1$. We call $f : S \to M$ a homology between $f_0 : S \to M$ and $f_1 : S \to M$.*

Let $f_0 : S_{0,\mu_0} \to M$ and $f_1 : S_{1,\mu_1} \to M$ be two immersed oriented measured solenoids. They are homology equivalent if there exists an immersed oriented measured solenoid $f : S_\mu \to M$ such that $f : S \to M$ is a homology between $f_0 : S \to M$ and $f_1 : S \to M$ and S_μ is a measured oriented cobordism from S_0 to S_1.

Clearly two homotopic immersions of a solenoid give homology equivalent immersions.

Theorem 3.8. *Suppose that two oriented measured solenoids $f_0 : S_{0,\mu_0} \to M$ and $f_1 : S_{1,\mu_1} \to M$ immersed in M are homology equivalent. Then the generalized currents coincide*

$$[f_0, S_{0,\mu_0}] = [f_1, S_{1,\mu_1}]\,.$$

The same happens if they are immersed equivalent.

Proof. In the first case, let ω be a closed k-form on M, then Stokes' theorem gives

$$\langle [f_1, S_{1,\mu_1}], \omega\rangle - \langle [f_0, S_{0,\mu_0}], \omega\rangle = \langle [f_{|\partial S}, \partial S_\mu], \omega\rangle = \langle [f, S_\mu], d\omega\rangle = 0\,.$$

In the second case, $f_0 = f_1 \circ h$ implies that the actions of the generalized currents over a closed form on M coincide, since the pull-back of the form to the solenoids agree through the diffeomorphism h, and the integrals over the transversal measure gives the same numbers, since the measures correspond by h. $\square$

Remark 3.9. In both definitions 3.5 and 3.6, we do not need to require that f be an immersion. Actually, the generalized current $[f, S_\mu]$ makes sense for any measured solenoid S_μ and any regular map $f : S \to M$, of class $C^{r,s}$ with $r \geq 1$. Theorem 3.8 still holds with these extended notions.

4. Intersection theory of solenoids

Let M be a smooth C^∞ oriented manifold.

Definition 4.1. (Transverse intersection) *Let $f_1 : S_1 \to M$, $f_2 : S_2 \to M$ be two immersed solenoids in M. We say that they intersect transversally if, for every $p_1 \in S_1$, $p_2 \in S_2$ such that $f_1(p_1) = f_2(p_2)$, the images of the leaves through p_1 and p_2 intersect transversally.*

If two immersed solenoids $f_1 : S_1 \to M$, $f_2 : S_2 \to M$, of dimensions k_1, k_2 respectively, intersect transversally, we define the intersection solenoid $f : S \to M$ as follows. The solenoid S is:

$$S = \{(p_1, p_2) \in S_1 \times S_2 \ ; \ f_1(p_1) = f_2(p_2)\} . \tag{1}$$

and the map $f : S \to M$ is given by

$$f(p_1, p_2) = f_1(p_1) = f_2(p_2), \quad (p_1, p_2) \in S . \tag{2}$$

We will see that S, the intersection solenoid, is indeed a solenoid. Also the intersection $f : S \to M$ of the two immersed solenoids $f_1 : S_1 \to M$, $f_2 : S_2 \to M$ is an immersed solenoid. In order to prove this, we consider the intersection of the product solenoid $F = f_1 \times f_2 : S_1 \times S_2 \to M \times M$ with the diagonal $\Delta \subset M \times M$. So we have to analyze first the case of the intersection of an immersed solenoid with a submanifold. The notion of transverse intersection given in definition 4.1 applies to this case (a submanifold is an embedded solenoid).

Lemma 4.2. *Let $f : S \to M$ be an immersed k-solenoid in M intersecting transversally an embedded closed submanifold $N \subset M$ of codimension q. Suppose that $S' = f^{-1}(N) \subset S$ is non-empty, then $f_{|S'} : S' \to M$ is an immersed $(k-q)$-solenoid in N.*

If S and N are oriented, so is S'.

If S has a transversal measure μ, then S' inherits a natural transversal measure, also denoted by μ.

Proof. First of all, note that S' is a compact and Hausdorff space.

Let W be a foliated manifold defining the solenoid structure of S such that there is a smooth map $\hat{f} : W \to M$ of class $C^{r,s}$, extending $f : S \to M$, which is an immersion on leaves. By definition, for any leaf $l \subset S$, $f(l)$ is transverse to N. Thus reducing W if necessary, the same transversality property occurs for any leaf of W. The transversality of the leaves implies that the map $\hat{f} : W \to M$ is transversal to the submanifold $N \subset M$, meaning that for any $p \in W$ such that $\hat{f}(p) \in N$,

$$d\hat{f}(p)(T_pW) + T_{\hat{f}(p)}N = T_{\hat{f}(p)}M .$$

This implies that $W' = \hat{f}^{-1}(N)$ is a submanifold of W of codimension q (in particular, $k - q \geq 0$). Moreover, it is foliated by the connected components l' of $l \cap \hat{f}^{-1}(N) = (\hat{f}_{|l})^{-1}(N)$, where

l are the leaves of W. By transversality of $\hat{f}$ along the leaves, l' is a $(k-q)$-dimensional submanifold of l. So W' is a foliated manifold with leaves of dimension $k-q$. This gives the required solenoid structure to $S' = S \cap \hat{f}^{-1}(N) = f^{-1}(N)$.

Clearly, $f_{|S'} : S' \to N$ is an immersion (of class $C^{r,s}$) since $\hat{f}_{|W'} : W' \to N$ is a smooth map which is an immersion on leaves.

If S and N are oriented, then each intersection $l' = l \cap \hat{f}^{-1}(N)$ is also oriented (using that M is oriented as well). Therefore the leaves of S' are oriented, and hence S' is an oriented solenoid.

Let $p \in S'$ and let $U \cong D^k \times K(U)$ be a flow-box for S around p. We can take U small enough so that $f(U)$ is contained in a chart of M in which N is defined by functions $x_1 = \ldots = x_q = 0$. By the transversality property, the differentials $dx_1, \ldots, dx_q$ are linearly independent on each leaf $f(D^k \times \{y\})$, $y \in K(U)$. Therefore, $x_1, \ldots, x_q$ can be completed to a set of functions $x_1, \ldots, x_k$ such that $dx_1, \ldots, dx_k$ are a basis of the cotangent space for each leaf (reducing U if necessary). Thus the pull-back of $x = (x_1, \ldots, x_k)$ to U give coordinates functions so that, using the coordinate $y \in K(U)$ for the transversal direction, (x, y) are coordinates for U, and $f^{-1}(N)$ is defined as $x_1 = \ldots = x_q = 0$. This means that

$$S' \cap U \cong \{(0, \ldots, 0, x_{q+1}, \ldots, x_k, y) \in D^k \times K(U)\} \cong D^{k-q} \times K(U)\,.$$

Therefore any local transversal T for S' is a local transversal for S, and any holonomy map for S' is a holonomy map for S. So

$$\mathrm{Hol}_{S'}(T) \subset \mathrm{Hol}_S(T)\,.$$

Hence a transversal measure for S gives a transversal measure for S'. □

Now we can address the general case.

Proposition 4.3. *Suppose that $f_1 : S_1 \to M$, $f_2 : S_2 \to M$ are two immersed solenoids in M intersecting transversally, and let S be its intersection solenoid defined in (1) and let f be the map (2). If $S \neq \emptyset$, then $f : S \to M$ is an immersed solenoid of dimension $k = k_1 + k_2 - n$ (in particular, k is a non-negative number).*

If S_1 and S_2 are both oriented, then S is also oriented.

If S_1 and S_2 are endowed with transversal measures μ_1 and μ_2 respectively, then S has an induced measure μ.

Proof. The product $S_1 \times S_2$ is a $(k_1 + k_2)$-solenoid and

$$F = f_1 \times f_2 : S_1 \times S_2 \to M \times M$$

is an immersion. Let $\Delta \subset M \times M$ be the diagonal. There is an identification (as sets)

$$S = (S_1 \times S_2) \cap F^{-1}(\Delta)\,.$$

The condition that $f_1 : S_1 \to M$, $f_2 : S_2 \to M$ intersect transversally can be translated into that $F, S_1 \times S_2 \to M \times M$ and Δ intersect transversally in $M \times M$.

Therefore applying lemma 4.2, $(S, F_{|S})$ is an immersed k-solenoid, where $F_{|S} : S \to \Delta$ is defined as $F(x_1, x_2) = f_1(x_1)$. Using the diffeomorphism $M \cong \Delta$, $x \mapsto (x, x)$, $F_{|S}$ corresponds to $f : S \to M$. So $f : S \to M$ is an immersed k-solenoid.

If S_1 and S_2 are both oriented, then $S_1 \times S_2$ is also oriented. By lemma 4.2, S inherits an orientation.

If S_1 and S_2 are endowed with transversal measures μ_1 and μ_2, then $S_1 \times S_2$ has a product transversal measure μ. For any local transversals T_1 and T_2 to S_1 and S_2, respectively, $T = T_1 \times T_2$ is a local transversal to $S_1 \times S_2$ (and conversely). We define

$$\mu_T = \mu_{1,T_1} \times \mu_{2,T_2} \,. \tag{3}$$

Now lemma 4.2 applies to give the transversal measure for S. Note that the local transversals to S are of the form $T_1 \times T_2$, for some local transversals T_1 and T_2 to S_1 and S_2. □

Remark 4.4. If $k_1 + k_2 = n$ then S is a 0-solenoid. For a 0-solenoid S, an orientation is a continuous assignment $\epsilon : S \to \{\pm 1\}$ of sign to each point of S.

Note also that for a 0-solenoid S, $T = S$ is a transversal and a transversal measure is a Borel measure on S.

Let $f_1 : S_1 \to M$, $f_2 : S_2 \to M$ be two immersed solenoids in M intersecting transversally, with $f : S \to M$ its intersection solenoid. Let $p = (p_1, p_2) \in S$. Then we can choose flow-boxes $U_1 = D^{k_1} \times K(U_1)$ for S_1 around p_1 with coordinates $(x_1, \ldots, x_{k_1}, y)$, and $U_2 = D^{k_2} \times K(U_2)$ for S_2 around p_2 with coordinates $(x_1, \ldots, x_{k_2}, z)$, and coordinates for M around $f(p)$, such that

$$\begin{aligned} f_1(x, y) &= (x_1, \ldots, x_{k_1+k_2-n}, x_{k_1+k_2-n+1}, \ldots, x_{k_1}, B_1(x, y), \ldots, B_{n-k_1}(x, y)) \,, \\ f_2(x, z) &= (x_1, \ldots, x_{k_1+k_2-n}, C_1(x, z), \ldots, C_{n-k_2}(x, z), x_{k_1+k_2-n+1}, \ldots, x_{k_2}) \,. \end{aligned}$$

Then S is defined locally as $D^{k_1+k_2-n} \times K(U_1) \times K(U_2)$ with coordinates $(x, y, z) = (x_1, \ldots, x_{k_1+k_2-n}, y, z)$ and

$$\begin{aligned} & f(x_1, \ldots, x_{k_1+k_2-n}, y, z) = \\ & \quad = (x_1, \ldots, x_{k_1+k_2-n}, C_1(x, z), \ldots, C_{n-k_2}(x, z), B_1(x, y), \ldots, B_{n-k_1}(x, y)) \,. \end{aligned}$$

Theorem 4.5. *Let $f : S_\mu \to M$ be an oriented measured k-solenoid immersed in M intersecting transversally a closed subvariety $i : N \hookrightarrow M$ of codimension q, such that $S' = f^{-1}(N) \subset S$ is non-empty. Consider the oriented measured $(k-q)$-solenoid immersed in N, $f' : S' \to M$, where $f' = f_{|S'}$. Then, under the restriction map*

$$i^* : H_c^{n-k}(M) \to H_c^{(n-q)-(k-q)}(N) \,, \tag{4}$$

the dual of the Ruelle-Sullivan homology class $[f, S_\mu]^*$ maps to $[f', S'_\mu]^*$.

Proof. Let $U \subset M$ be a tubular neighbourhood of N with projection $\pi : U \to N$. Note that U is diffeomorphic to the unit disc bundle of the normal bundle of N in M. Let τ be a Thom form for $N \subset M$, that is a closed form $\tau \in \Omega^q(M)$ supported in U, whose integral in any normal space $\pi^{-1}(n)$, $n \in N$, is one. The dual of the map (4) under Poincaré duality is the map

$$H^{k-q}(N) \to H^k(M) \,,$$

which sends $[\beta] \in H^{k-q}(N)$ to $[\tilde{\beta}]$, where $\tilde{\beta} = \pi^* \beta \wedge \tau$ (this form is extended from U to the whole of M by zero). So we only need to see that

$$\langle [f, S_\mu], \tilde{\beta} \rangle = \langle [f', S'_\mu], \beta \rangle \,.$$

Take a covering of S by flow-boxes $U_i \cong D^k \times K(U_i) \cong D^q \times D^{k-q} \times K(U_i)$ so that $U'_i = U_i \cap S'$ is given by $x_1 = \ldots = x_q = 0$. Making the tubular neighborhood $U \supset N$ smaller if necessary, we can arrange that $f^{-1}(U) \cap U_i$ is contained in $D^q_r \times D^{k-q} \times K(U_i)$, for some $r < 1$. It is easy to construct a map $\tilde{\pi} : f^{-1}(U) \to f^{-1}(N)$ which consists on projecting in the normal directions along the leaves. Then $f \circ \tilde{\pi}$ and $\pi \circ f$ are homotopic.

Let S'_i be a measurable partition of S' with $S'_i \subset U'_i$. We may assume that $S_i = \tilde{\pi}^{-1}(S'_i)$ is contained in U_i. The sets S_i form a measurable partition containing $f^{-1}(U)$, the support of $f^*\tilde{\beta} = f^*(\pi^*\beta \wedge \tau)$. Then

$$\begin{aligned}
\langle [f, S_\mu], \tilde{\beta}\rangle &= \sum_i \int_{K(U_i)} \left(\int_{S_i \cap (D^k \times \{y\})} f^*(\pi^*\beta \wedge \tau) \right) d\mu_{K(U_i)}(y) \\
&= \sum_i \int_{K(U_i)} \left(\int_{S_i \cap (D^k \times \{y\})} \tilde{\pi}^* f^*\beta \wedge f^*\tau) \right) d\mu_{K(U_i)}(y) \\
&= \sum_i \int_{K(U_i)} \left(\int_{S'_i \cap (D^{k-q} \times \{y\})} f^*\beta \right) \left(\int_{f(D^q)} \tau \right) d\mu_{K(U_i)}(y) \\
&= \sum_i \int_{K(U'_i)} \left(\int_{S'_i \cap (D^{k-q} \times \{y\})} f^*\beta \right) d\mu_{K(U'_i)}(y) \\
&= \langle [f', S'_\mu], \beta \rangle \, .
\end{aligned}$$

□

Theorem 4.6. *Suppose that $f_1 : S_{1,\mu_1} \to M$, $f_2 : S_{2,\mu_2} \to M$ are two oriented measured immersed solenoids in M intersecting transversally, and let $f, S_\mu \to M$ be its intersection solenoid. Then the duals of the Ruelle-Sullivan homology classes satisfy*

$$[f, S_\mu]^* = [f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^* \, .$$

Proof. Note that $[f_1, S_{1,\mu_1}]^* \in H_c^{n-k_1}(M)$ and $[f_2, S_{2,\mu_2}]^* \in H_c^{n-k_2}(M)$, so $[f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^*$ and $[f, S]^*$ both live in

$$H_c^{n-k_1+n-k_2}(M) = H_c^{n-k}(M) \, .$$

Consider the immersed solenoid $(F, S_1 \times S_2)$, where $F = f_1 \times f_2 : S_1 \times S_2 \to M \times M$ and $S_1 \times S_2$ has the transversal measure μ given by (3). Let us see that the following equality, involving the respective generalized currents,

$$[F, (S_1 \times S_2)_\mu] = [f_1, S_{1,\mu_1}] \otimes [f_2, S_{2,\mu_2}] \in H_{k_1+k_2}(M \times M)$$

holds. We prove this by applying both sides to $(k_1 + k_2)$-cohomology classes in $M \times M$. Using the Künneth decomposition it is enough to evaluate on a form $\beta = p_1^*\beta_1 \wedge p_2^*\beta_2$, where $\beta_1, \beta_2 \in H^*(M)$ are closed forms and $p_1, p_2 : M \times M \to M$ are the two projections. Let $\{U_i\}$, $\{V_j\}$ be open covers of S_1, S_2 respectively, by flow-boxes, and let $\{\rho_{1,i}\}$, $\{\rho_{2,j}\}$ be partitions of unity subortinated to such covers. Then

$$\begin{aligned}
&\langle [F, (S_1 \times S_2)_\mu], \beta \rangle = \\
&= \sum_{i,j} \int_{K(U_i) \times K(V_j)} \left(\int_{L_y \times L_z} (p_1^*\rho_{1,i}) \, (p_2^*\rho_{2,j}) F^*(p_1^*\beta_1 \wedge p_2^*\beta_2) \right) d\mu_{K(U_i) \times K(V_j)}(y, z) \\
&= \sum_{i,j} \int_{K(U_i) \times K(V_j)} \left(\int_{L_y \times L_z} p_1^*(\rho_{1,i} f_1^*\beta_1) \wedge p_2^*(\rho_{2,j} f_2^*\beta_2) \right) d\mu_{1,K(U_i)}(y) \, d\mu_{2,K(V_j)}(z) \\
&= \left(\sum_i \int_{K(U_i)} \left(\int_{L_y} \rho_{1,i} f_1^*\beta_1 \right) d\mu_{1,K(U_i)}(y) \right) \left(\sum_j \int_{K(V_j)} \left(\int_{L_z} \rho_{1,j} f_2^*\beta_2 \right) d\mu_{2,K(V_j)}(y) \right) \\
&= \langle [f_1, S_{1,\mu_1}], \beta_1 \rangle \, \langle [f_2, S_{2,\mu_2}], \beta_2 \rangle \, ,
\end{aligned}$$

as required.

Now we are ready to prove the statement of the theorem. Let $\varphi : M \to \Delta$ be the natural diffeomorphism of M with the diagonal $\Delta \subset M \times M$, and let $i : \Delta \hookrightarrow M \times M$ be the inclusion. Then, using theorem 4.6,

$$\begin{aligned} [f, S_\mu]^* &= [\varphi \circ f, S_\mu]^* = i^*([F, (S_1 \times S_2)_\mu]^*) = \\ &= i^*([f_1, S_{1,\mu_1}]^* \otimes [f_2, S_{2,\mu_2}]^*) = [f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^* . \end{aligned}$$

□

Let us look more closely to the case where $k_1 + k_2 = n$. We assume that $f_1 : S_{1,\mu_1} \to M$, $f_2 : S_{2,\mu_2} \to M$ are two oriented immersed measured solenoids of dimensions k_1, k_2 respectively, which intersect transversally. Let $f : S_\mu \in M$ be the intersection 0-solenoid of $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$.

Definition 4.7. (Intersection index) *At each point $x = (x_1, x_2) \in S$, the intersection index $\epsilon(x_1, x_2) \in \{\pm 1\}$ is the sign of the intersection of the leaf of S_1 through x_1 with the leaf of S_2 through x_2. The continuous function $\epsilon : S \to \{\pm 1\}$ gives the orientation of S.*

Recall that the 0-solenoid $f : S_\mu \to M$ comes equipped with a natural measure μ (for a 0-solenoid the notions of measure and transversal measure coincide). If $x = (x_1, x_2) \in S$, then locally around x, S is homeomorphic to $T = T_1 \times T_2$, where T_1 and T_2 are small local transversals of S_1 and S_2 at x_1 and x_2, respectively. The measure μ_T is the product measure $\mu_{1,T_1} \times \mu_{2,T_2}$.

Definition 4.8. (Intersection measure) *The intersection measure is the transversal measure μ of the intersection solenoid $f : S_\mu \to M$, induced by those of $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$.*

Definition 4.9. (Intersection pairing) *We define the intersection pairing as the real number*

$$(f_1, S_{1,\mu_1}) \cdot (f_2, S_{2,\mu_2}) = \int_S \epsilon \, d\mu .$$

Theorem 4.10. *If $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$ are two oriented immersed measured solenoids of dimensions k_1, k_2 respectively, which intersect transversally, such that $k_1 + k_2 = n$. Then*

$$(f_1, S_{1,\mu_1}) \cdot (f_2, S_{2,\mu_2}) = [f_1, S_{1,\mu_1}]^* \cdot [f_2, S_{2,\mu_2}]^* .$$

Proof. By theorem 4.6,

$$[f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^* = [f, S_\mu]^* \in H^n_c(M, \mathbb{R}) .$$

The intersection product $[f_1, S_{1,\mu_1}]^* \cdot [f_2, S_{2,\mu_2}]^*$ is obtained by evaluating this cup product on the element $1 \in H^0(M, \mathbb{R})$, i.e.

$$[f_1, S_{1,\mu_1}]^* \cdot [f_2, S_{2,\mu_2}]^* = \langle [f, S_\mu], 1 \rangle = \int_S f^*(1) d\mu(x) = \int_S \epsilon \, d\mu ,$$

since the pull-back of a function gets multiplied by the orientation of S, which is the function ϵ. □

When the solenoids are uniquely ergodic we can sometimes recover this intersection index by a natural limiting procedure. Recall that we say that a Riemannian solenoid S is of controlled growth (see definition 3.3 in [2]) if there is a leaf $l \subset S$, a point $p \in l$ such that the Riemannian balls $l_n \subset l$, of some radius $R_n \to \infty$, satisfy that for each flow-box U in a finite covering of S

the number of incomplete horizontal discs in $U \cap l_n$ is negligeable with respect to the number of complete horizontal discs in $U \cap l_n$. Then, if μ_n is the normalized measure corresponding to l_n, the limit $\mu = \lim_{n\to\infty} \mu_n$ is the unique Schwartzman measure (corollary 3.7 in [2]).

Theorem 4.11. *Let $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$ be two immersed, oriented, uniquely ergodic solenoids with controlled growth transversally intersecting. Let $l_1 \subset S_1$ and $l_2 \subset S_2$ be two arbitrary leaves. Choose two base points $x_1 \in l_1$ and $x_2 \in l_2$, and fix Riemannian exhaustions $(l_{1,n})$ and $(l_{2,n})$. Define*

$$(f_1, l_{1,n}) \cdot (f_2, l_{2,n}) = \frac{1}{M_n} \sum_{\substack{p=(p_1,p_2)\in l_{1,n}\times l_{2,n} \\ f_1(p_1)=f_2(p_2)}} \epsilon(p)\,,$$

where $M_n = \mathrm{Vol}_{k_1}(l_{1,n}) \cdot \mathrm{Vol}_{k_2}(l_{2,n})$.

Then

$$\lim_{n\to+\infty} (f_1, l_{1,n}) \cdot (f_2, l_{2,n}) = (f_1, S_{1\mu_1}) \cdot (f_2, S_{2,\mu_2})\,.$$

In particular, the limit exists and is independent of the choices of l_1, l_2, x_1, x_2 and the radius of the Riemannian exhaustions.

Proof. The key observation is that because of the unique ergodicity, the atomic transversal measures associated to the normalizad k-volume of the Riemannian exhaustions (name them $\mu_{1,n}$ and $\mu_{2,n}$) are converging to μ_1 and μ_2, respectively. In particular, in each local flow-box we have

$$\mu_{1,n} \times \mu_{2,n} \to \mu_1 \times \mu_2 = \mu\,.$$

Therefore the average defining $(f_1, l_{1,n}) \cdot (f_2, l_{2,n})$ converges to the integral defining $(f_1, S_{1\mu_1}) \cdot (f_2, S_{2,\mu_2})$ since ϵ is a continuous and integrable function (indeed bounded by 1). □

Remark 4.12. The previous theorem and proof work in the same form for ergodic solenoids, provided that we know that the Schwartzman limit measure for almost all leaves is the given ergodic measure. This is simple to prove for ergodic solenoids with trapping regions mapping to a contractible ball in M (cf. theorem 7.12 in [2]).

We end up this section with a perturbation result. We want to prove that we can achieve transversality for a large class of solenoids by a suitable homotopy. The solenoids that we have in mind are those whose transversal is a Cantor set.

Theorem 4.13. *Let $f : S \to M$ be a solenoid whose transversals are Cantor sets, and let $N \subset M$ be a smooth closed submanifold. Then we can homotop $f : S \to M$ so that N and S intersect transversely.*

Proof. Let $f : S \to M$ be an immersion of the solenoid into a manifold M. Recall that this means that the diferential of f along leaves is injective.

Let p be a point in the solenoid. We want to perturb f in a neighbourhood of p. Consider a flow-box $U \subset S$ of the form $D^k_{1+r} \times T$, where T is a Cantor set, and D^k_{1+r} is a k-disc of radius $1+r$, for some small real number $r > 0$. Consider also a coordinate chart $(x_1, \ldots, x_n)$ for M so that N is given by $x_1 = \ldots = x_q = 0$, and $p = (0, y_0)$. Define the composition

$$f : D^k_{1+r} \times T \to V \to \mathbb{R}^p\,.$$

The transversality of the leaf $L_y = D^k_1 \times \{y\}$ to N is equivalent to the transversality of the map $f_y := f(\,\cdot\,, y)$ to zero.

For any $\epsilon > 0$ small enough, there is a vector $v \in \mathbb{R}^p$ so that f_{y_0} is transversal to $-v$. Therefore $f_{y_0} + v$ is transversal to zero (that is, if $f_{y_0}(x) + v = 0$ then $df_{y_0}(x)$ is surjective). Moreover, there is an open neighbourhood of y_0, $T_0 \subset T$, where this transversality still holds.

Now take a bump function $\rho(x)$ which is one over D_1^k and it is zero near the boundary of D_{1+r}^k. The map

$$\hat{f}(x,y) = \begin{cases} f(x,y) + \rho(x)v, & x \in D_{1+r}^k, y \in T_0 \\ f(x,y), & \text{otherwise} \end{cases}$$

is smooth (here it is where we use that T is a Cantor set), transversal to N along $D_1^k \times T_0$.

Repeating this process, we can find a finite cover $T = \sqcup T_j$, and define the perturbations indepedently on $D_{1+r}^k \times T_j$.

Finally, we have managed to achieve transversality on $V = D_1^k \times T$, perturbing on $D_{1+r}^k \times T$. What we do now is to use a finite cover of S with subsets as V, and perturb successively. At each step we take a perturbation of norm small enough so that this does not destroy the perturbation over the set where it was previously achieved.

□

Remark 4.14. We can construct an example where it is not possible to perturb two solenoids (at least in a differentiable way) with transversal Cantor sets so that they intersect transversally.

Consider $M = \mathbb{R}^2$, and let $K_1, K_2 \subset [0,1]$ be two Cantor sets. Let S_1 be given by the leaves (x,y), $x \in \mathbb{R}$, $y \in K_1$. Let S_2 be given by the leaves (x, x^2+z), $x \in \mathbb{R}$, $z \in K_2$. These two solenoids intersect non-transversally at the points determined by $x = 0$, $y = z \in K_1 \cap K_2$.

Suppose that we have small perturbations S_1', S_2' of S_1, S_2, respectively. Then S_1' is defined by leaves of the form $(x, y + f_1(x,y))$, $x \in \mathbb{R}$, $y \in K_1$, and S_2' by leaves of the form $(x, x^2 + z + f_2(x,z))$, $x \in \mathbb{R}$, $z \in K_2$, where f_j is a smooth function on $\mathbb{R} \times K_j$, having small norm, $j = 1, 2$ (recall that a smooth function on $\mathbb{R} \times K_j$ extends as a smooth function on some neighbourhood of it). Composing with a suitable diffeomorphism of $\mathbb{R}^2$, we can suppose that $f_1 = 0$. So we are looking for non-transversal intersections of $S_1' = \{(x,y) \,|\, x \in \mathbb{R}, y \in K_1\}$ and $S_2' = \{(x, x^2 + z + g(x,z)) \,|\, x \in \mathbb{R}, z \in K_2\}$, g some small smooth function on $\mathbb{R} \times K_2$. These are obtained by solving

$$2x + g_x(x,z) = 0, \qquad y = x^2 + z + g(x,z)\,.$$

The equation $2x + g_x(x,z) = 0$ can be solved as $x = \phi(z)$, for some (small) smooth function ϕ. Write $r(z) = \phi(z)^2 + z + g(\phi(z), z)$, which is a smooth function on K_2 close to $r_0(z) = z$. This defines a smooth isotopy of K_2. Let $K_2' = r(K_2)$. The points of non-transversal intersections of S_1' and S_2' are given by solving $y = r(z)$, so they correspond to the points in $K_1 \cap K_2'$.

To guarantee that $K_1 \cap K_2' \neq \emptyset$, just choose K_1, K_2 two Cantor sets with positive Lebesgue measure such that $\mu(K_1) + \mu(K_2) > 1$. As K_2' is a small smooth perturbation of K_2, the measure of K_2' is close to that of K_2. So K_1 and K_2' must intersect.

5. Almost everywhere transversality

The intersection theory developed in section 4 is not fully satisfactory since there are examples of solenoids (e.g. foliations) which do not intersect transversally, and cannot be perturbed to do so. Even in the case of solenoids whose transversal structure is Cantor, sometimes it is not possible to perturb smoothly the solenoid to make them intersect transversally, as remark

4.14 shows. Another example is given by the persistence of homoclinic tangencies for stable and unstable foliations shows (see [5]).

However, a weaker notion is enough to develop intersection theory for solenoids. Indeed, the intersection pairing can also be defined for oriented, measured solenoids $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$, immersed in an oriented n-manifold M, with $k_1 + k_2 = n$, $k_1 = \dim S_1$, $k_2 = \dim S_2$, which intersect transversally almost everywhere in the following sense:

Definition 5.1. **(Almost everywhere transversality)** *Let $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$ be two measured immersed oriented solenoids. They intersect almost everywhere transversally if the set*

$$F = \{(p_1, p_2) \in S_1 \times S_2\,; \\ f_1(p_1) = f_2(p_2), df_1(p_1)(T_{p_1}S_1) + df_2(p_2)(T_{p_2}S_2) \neq T_{f_1(p_1)}M\} \subset S_1 \times S_2$$

of non-transversal intersection points satisfies:

(1) *every point $p \in F$ is an isolated point of*

$$S = \{(p_1, p_2) \in S_1 \times S_2\,;\, f_1(p_1) = f_2(p_2)\} \subset S_1 \times S_2$$

in the leaf of $S_1 \times S_2$ through p.

(2) *F is null-transverse in $S_1 \times S_2$ (with the natural product transversal measure μ), i.e. if the set of leaves of $S_1 \times S_2$ intersecting F has zero μ-measure.*

It is useful to translate to $S_1 \times S_2$ the meaning of almost everwywhere transversality.

Definition 5.2. **(Almost everywhere transversality)** *Let $f : S_\mu \to M$ be a measured immersed oriented k-solenoid and $N \subset M$ a closed submanifold of codimension k. They intersect almost everywhere transversally if the set*

$$F = \{p \in S\,;\, f(p) \in N,\, df(p)(T_pS) + T_{f(p)}N \neq T_{f(p)}M\} \subset S$$

of non-transversal intersection points satisfies that:

(1) *every point $p \in F$ is isolated as a point of S' in the leaf of S through p,*
(2) *$F \subset S_\mu$ is null-transverse, i.e. for any flow-box $U = D^k \times K(U)$, the projection by $\pi : U = D^k \times K(U) \to K(U)$ of the intersection $F \cap U$, that is $\pi(F \cap U) \subset K(U)$, is of zero $\mu_{K(U)}$-measure in $K(U)$.*

Note that a set $F \subset S_\mu$ in a measured solenoid is null-transverse if for any local transversal T, the set of leaves passing through F intersects T in a set of zero μ_T-measure.

Every point of $S' - F$ is automatically isolated as a point of S' in the leaf of S through it. Therefore condition (1) is equivalent to saying that every point of S' is isolated in the corresponding leaf.

Then we have the following straightforward lemma.

Lemma 5.3. *The solenoids $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$ are almost everywhere transversal if and only if $f_1 \times f_2 : (S_1 \times S_2)_\mu \to M \times M$ and the diagonal $\Delta \subset M \times M$ intersect almost everywhere transversally.*

Let $f : S_\mu \to M$ be an immersed solenoid intersecting transversally almost everywhere a closed submanifold $N \subset M$. Write $S' = f^{-1}(N)$ and let $F \subset S'$ be the subset of non-transversal points. Note that $S'_{reg} = S' - F$ is open in S' and F is closed. Moreover, S'_{reg}

consists of the transversal intersections, so the intersection index $\epsilon : S'_{reg} \to \{\pm 1\}$ is well defined and continuous. We define the intersection number as

$$\int_{S'-F} \epsilon(x) d\mu(x) .$$

Theorem 5.4. *Suppose that an immersed measured oriented k-solenoid $f : S_\mu \to M$ and a submanifold $N \subset M$ of codimension k intersect almost everywhere transversally. Then*

$$[f, S_\mu]^* \cdot [N] = \int_{S'-F} \epsilon \, d\mu .$$

Proof. Fix an accessory Riemannian metric on M. By pull-back, this gives a metric on S.

Let $p \in F$. Then by assumption, there is some $\eta > 0$ such that

$$B_{\frac{3}{2}\eta}(p) \cap F = B_{\frac{3}{2}\eta}(p) \cap S' = \{p\},$$

where $B_r(p)$ is the Riemannian ball in the leaf centered at p and of radius $r > 0$. It is easy to construct a flow-box $U = D^k \times K(U)$ with $p = (0, y_0) \in U$ so that

(i) $D^k \times \{y_0\} = B_{\frac{3}{2}\eta}(p)$,
(ii) $D^k_{3/4} \times \{y_0\} = B_\eta(p)$, ($D^k_r$ denotes the open disc of radius $r > 0$),
(iii) the open annulus $A = (D^k - \bar{D}^k_{1/2})$ satisfies that $(A \times K(U)) \cap S' = \emptyset$,
(iv) the intersection number $[f(D^k_{3/4} \times \{y\})] \cdot [N]$ is constant for $y \in K(U)$.

For achieving this, take $K(U)$ small enough. Note that the intersection number in (iv) is well-defined since $f(\partial(D^k_{3/4}) \times \{y\})$ does not touch N; and it is locally constant by continuity. We fix a finite covering $\{U_i\}$ of F with such flow-boxes.

Let $\pi_i : U_i = D^k \times K(U_i) \to K(U_i)$ be the projection onto the second factor. By hypothesis, $\pi_i(F \cap U_i)$ is of zero measure. We may take a nested sequence $(V_{i,n})$ of open neighbourhoods of $\pi_i(F \cap U_i)$ in $K(U_i)$ such that $\bigcap_{n \geq 1} V_{i,n} = \pi_i(F \cap U_i)$. Let

$$U_{i,n} = D^k_{3/4} \times V_{i,n}$$

and

$$U_n = \bigcup_i U_{i,n} .$$

Then (U_n) is a nested sequence of open neighbourhoods of F in S. It may happen that $\bigcap_{n \geq 1} U_n$ contains points of $S' - F$, but this is a set of μ-measure zero. So

$$\int_{S'-U_n} \epsilon \, d\mu \longrightarrow \int_{S' - \bigcap_{n\geq 1} U_n} \epsilon \, d\mu = \int_{S'-F} \epsilon \, d\mu .$$

As $S' - U_n$ is compact, the angle of intersection in $S' - U_n$ between $f(S)$ and N is bounded below, so there is a small $\rho > 0$ (depending on n) such that if U_ρ is the ρ-tubular neighbourhood of N in M, then for each intersection point $x \in S' - U_n$, there is a (topological) disc D_x contained in a local leaf through x, which is exactly the path component of $f^{-1}(U_\rho)$ through x. Making ρ smaller we can assume that D_x is as small as we want. Note that (iii) guarantees that D_x does not touch $D^k_{3/4} \times V_{i,n} = U_{i,n}$ for any i. So $D_x \subset S - U_n$.

Let τ_ρ be a Thom form for $N \subset M$, that is a closed k-form supported in U_ρ, whose integral in the normal space to N is one. Then $\int_{D_x} \tau_\rho = 1$ for any $x \in S' - U_n$. So

$$\int_{S'-U_n} \epsilon \, d\mu = \int_{S-U_n} f^* \tau_\rho .$$

On the other hand,

$$\int_{U_{i,n}} f^*\tau_\rho = \int_{V_{i,n}} \left(\int_{D^k_{3/4}\times\{y\}} f^*\tau_\rho \right) d\mu_{K(U_i)} \le C\, \mu_{K(U_i)}(V_{i,n}) \to 0\,,$$

where C is a bound for all the intersection numbers in (iv) for all U_i simultaneously. Then

$$\int_{U_n} f^*\tau_\rho \to 0\,,$$

when $n\to\infty$.

Putting everything together,

$$\begin{aligned} [f,S_\mu]^*\cdot[N] &= \langle [f,S_\mu],[\tau_\rho]\rangle = \int_{S_\mu} f^*\tau_\rho = \int_{S-U_n} f^*\tau_\rho + \int_{U_n} f^*\tau_\rho \\ &= \int_{S'-U_n} \epsilon\, d\mu + \int_{U_n} f^*\tau_\rho \to \int_{S'-F} \epsilon\, d\mu\,. \end{aligned}$$

□

Remark 5.5. It is not true that, without further restrictions, the measure μ is finite on $S'_{reg} = S'-F$. For instance, it may happen that around a point $p\in F$, there are leaves of S (leaves which do not go through F) with arbitrary large number of positive and negative intersections with N (near p). Obviously, the difference between positive and negative intersections is bounded. Therefore, there is no current associated to (S'_{reg},μ).

Consider now two immersed measured oriented solenoids $f_1 : S_{1,\mu_1}\to M$, $f_2 : S_{2,\mu_2}\to M$ intersecting almost everywhere transversally. Let $F\subset S_1\times S_2$ be the subspace of non-transversal intersection points, which has null-transversal measure in $S_1\times S_2$. Set $S=(S_1\times S_2)\cap \tilde f^{-1}(\Delta)$, where $\tilde f = f_1\times f_2$. Then there is an intersection index $\epsilon(x)$ for each $x\in S-F$ and an intersection measure μ on $S-F$. We define the intersection product as

$$\int_{S-F} \epsilon\, d\mu\,.$$

Then theorem 5.4 implies the following:

Theorem 5.6. *In the situation above, we have that*

$$[f_1,S_{1,\mu_1}]^*\cdot[f_2,S_{2,\mu_2}]^* = \int_{S-F} \epsilon(x)\, d\mu(x)\,.$$

6. Intersection of analytic solenoids

It is now our intention to translate the theory of solenoids intersecting almost-everywhere transversally to the case where the dimensions are not complementary, that is, when $k_1+k_2>n$.

Definition 6.1. (Almost everywhere transversality) *Let $f_1 : S_{1,\mu_1}\to M$ and $f_2 : S_{2,\mu_2}\to M$ be two measured immersed oriented solenoids in an oriented n-manifold M, with $k_1+k_2\ge n$, $k_1=\dim S_1$, $k_2=\dim S_2$. They intersect almost everywhere transversally if the set*

$$F=\{(p_1,p_2)\in S_1\times S_2\,;\; f_1(p_1)=f_2(p_2), df_1(p_1)(T_{p_1}S_1)+df_2(p_2)(T_{p_2}S_2)\neq T_{f_1(p_1)}M\}\subset S_1\times S_2$$

of non-transversal intersection points satisfies:

(1) *every point $p\in F$ is an isolated point of F in the leaf of $S_1\times S_2$ through p.*

(2) *the set*
$$S = \{(p_1, p_2) \in S_1 \times S_2\,;\, f_1(p_1) = f_2(p_2)\} \subset S_1 \times S_2$$
is $C^{1,0}$-conjugate, locally near any $p \in F$, to a leafwise (real) analytic set (i.e. it is of class $C^{\omega,0}$).

(3) *F is null-transverse in $S_1 \times S_2$ (with the natural product transversal measure μ), i.e. if the set of leaves of $S_1 \times S_2$ intersecting F has zero μ-measure.*

Then it is useful to translate to $S_1 \times S_2$ the meaning of almost everwywhere transversality.

Definition 6.2. (Almost everywhere transversality) *Let $f : S_\mu \to M$ be a measured immersed oriented k-solenoid and $N \subset M$ a closed submanifold of codimension q. They intersect almost everywhere transversally if the set*
$$F = \{p \in S\,;\, f(p) \in N,\, df(p)(T_pS) + T_{f(p)}N \neq T_{f(p)}M\} \subset S$$
of non-transversal intersection points satisfies that:

(1) *every point $p \in F$ is isolated as a point of F in the leaf of S through p,*

(2) *the set*
$$S' = \{p \in S\,;\, f(p) \in N\} \subset S$$
is $C^{1,0}$-conjugate, locally near any $p \in F$, to a leafwise (real) analytic set (i.e. it is of class $C^{\omega,0}$).

(2) *$F \subset S_\mu$ is null-transverse, i.e. for any flow-box $U = D^k \times K(U)$, the projection by $\pi : U = D^k \times K(U) \to K(U)$ of the intersection $F \cap U$, that is $\pi(F \cap U) \subset K(U)$, is of zero $\mu_{K(U)}$-measure in $K(U)$.*

We have the following straightforward lemma.

Lemma 6.3. *The solenoids $f_1 : S_{1,\mu_1} \to M$ and $f_2 : S_{2,\mu_2} \to M$ are almost everywhere transversal if and only if $\tilde{f} = f_1 \times f_2 : (S_1 \times S_2)_\mu \to M \times M$ and the diagonal $\Delta \subset M \times M$ intersect almost everywhere transversally.*

Let $f : S_\mu \to M$ be an immersed oriented k-solenoid intersecting transversally almost everywhere a closed oriented submanifold $N \subset M$ of codimension q. Write $S' = f^{-1}(N)$ and let $F \subset S'$ be the subset of non-transversal points. Note that $S'_{reg} = S' - F$ is open in S' and F is closed. Let $p \in S'_{reg} = S' - F$. Then the transversal intersection property implies that there exists a flow-box $U = D^k \times K(U)$ for S around p, with coordinates $(x_1, \ldots, x_k, y_1, \ldots, y_l)$ such that N is defined by $x_1 = \ldots = x_q = 0$. Thus S'_{reg} has locally the structure of $(k-q)$-dimensional oriented solenoid, with a transversal measure μ induced by μ and invariant by holonomy. Note that S'_{reg} is not a solenoid because it is not compact.

Theorem 6.4. *Suppose that a k-solenoid $f : S_\mu \to M$ and a submanifold $N \subset M$ of codimension q intersect almost everywhere transversally. Let $\omega \in \Omega^{k-q}(M)$ be a closed form. Then*
$$\langle [f, S_\mu]^* \cup [\omega], [N] \rangle = \int_{S'-F} \omega\,.$$

Proof. Let us take a Thom form τ_ρ for N, and consider the current
$$(f, S_\mu) \wedge \tau_\rho \qquad (5)$$
defined as the wedge of the generalized current with the smooth form τ_ρ. Let us see that there is a limit for (5) when $\rho \to 0$.

Let us define the current of integration S'_{reg}. This is obviously well-defined off F. Now suppose that ω is a $(k-q)$-form supported in a small ball around a point $p \in F$. Let $U = D^k \times K(U)$ be a flow-box around p, where ω is defined. Then, after taking a $C^{1,0}$-diffeomorphism, we can suppose that S' is defined as $f(x,y) = 0$, for some $f : D^k \times K(U) \to \mathbb{R}^q$ of class $C^{\omega,0}$. Then $S_y = S'_{reg} \cap L_y = f_y^{-1}(0)$ is an analytic subset. Hence the integral of ω on S_y is bounded

$$\int_{S_y} \omega \leq C||\omega|| \, ,$$

where C is a constant that we can suppose valid for all $y \in K(U)$ by continuity.

Moreover, making the radius of the ball smaller, we have that $C \to 0$. Hence

$$\langle (S_{reg}), \omega \rangle := \int_{K(U)} \left(\int_{S_y} \omega \right) d\mu_{K(U)}(y)$$

is well-defined.

Now, to see that

$$(f, S_\mu) \wedge \tau_\rho \to (S_{reg}) \, ,$$

we apply both sides to a $(k-q)$-form ω. For ω supported in a flow-box off F, we have that

$$\langle (f, S_\mu) \wedge \tau_\rho, \omega \rangle = \int_{K(U)} \int_{L_y} \tau_\rho \wedge \omega \, d\mu_{K(U)}(y) \to \int_{K(U)} \int_{S_y} \omega \, d\mu_{K(U)}(y) = \langle (S_{reg}), \omega \rangle \, .$$

Now let ω supported in an ϵ-ball around $p \in F$. Then $\langle (S_{reg}), \omega \rangle$ is as small as we want, and

$$\lim_{\rho \to 0} \langle (f, S_\mu) \wedge \tau_\rho, \omega \rangle = \lim_{\rho \to 0} \int_{K(U)} \int_{L_y} \tau_\rho \wedge \omega$$

is small. Since $\mu_{K(U)}(K(U))$ is small, it only remains to see that

$$\lim_{\rho \to 0} \int_{L_y} \tau_\rho \wedge \omega = \int_{S_y} \omega$$

is bounded for y off the bad locus. This is bounded by the area of S_y times the norm of ω, and both these quantities are bounded (the first one is bounded due to the transversal continuity).

□

Consider now two immersed measured oriented solenoids $f_1 : S_{1,\mu_1} \to M$, $f_2 : S_{2,\mu_2} \to M$ intersecting almost everywhere transversally. Let $F \subset S_1 \times S_2$ be the subspace of non-transversal intersection points, which has null-transversal measure in $S_1 \times S_2$. Set $S = (S_1 \times S_2) \cap \tilde{f}^{-1}(\Delta)$, where $\tilde{f} = f_1 \times f_2 : S_1 \times S_2 \to M \times M$. Then Theorem 6.4 implies that

$$\langle [f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^*, [\omega] \rangle = \int_{S-F} \omega \, ,$$

for any closed form ω of degree $k_1 + k_2 - n$.

Corollary 6.5. *Let M be an analytic manifold, and let $f_1 : S_{1,\mu_1} \to M$, $f_2 : S_{2,\mu_2} \to M$ be two immersed measured oriented solenoids of class $C^{\omega,0}$ (that is, with analytic leaves). Let $S = \{(p_1, p_2) \in S_1 \times S_2 \, ; \, f_1(p_1) = f_2(p_2)\}$, and let $F \subset S$ consist of points (p_1, p_2) such that the leaves of S_1 and S_2 at p_1 and p_2 do not intersect transversally. Suppose that*

1. *every point $p \in F$ is an isolated point of F in the leaf of $S_1 \times S_2$ through p.*
2. *F is null-transverse in $S_1 \times S_2$.*

Then

$$\langle [f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^*, [\omega] \rangle = \int_{S-F} \omega \,,$$

for any $[\omega] \in H^{k_1+k_2-n}(M)$.

Proof. We only need to note that condition (2) in Definition 6.1 is automatic. □

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Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Consejo Superior de Investigaciones Científicas, Serrano 113 bis, 28006 Madrid, Spain

Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain

E-mail address: vicente.munoz@imaff.cfmac.csic.es

CNRS, LAGA UMR 7539, Université Paris XIII, 99, Avenue J.-B. Clément, 93430-Villetaneuse, France

E-mail address: ricardo@math.univ-paris13.fr